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The average stress in incompressible disperse flow ☆

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Abstract

An analysis of the average stress in a disperse flow consisting of equal spherical particles suspended in a fluid is presented. Other than incompressibility, no assumptions are made on the rheological nature of the fluid. In particular, the Reynolds number of the particle motion relative to the fluid is arbitrary. The use of ensemble averages permits the consideration of spatially non-uniform systems, which reveals features not identified before. In particular, it is shown that, in general, the average stress is not symmetric, even when there are no external couples acting on the particles. A quantity to be identified with the mixture pressure (including the particle contribution) is identified. The structure of the momentum equations for the fluid and particle phases is systematically derived. As an example, the case of particles suspended in a locally Stokes flow is considered.
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1. Introduction

At the spatial and temporal scales of interest for most practical applications, the principal tool for the analysis of most multiphase flows must rest on some form of averaged mathematical description. While this basic fact has been widely recognized for many decades, progress in the

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formulation of such averaged models has been slow and has met with unexpected difficulties, not only with respect to physical realism, but also at the more basic level of the mathematical structure of the equations.

As in single-phase turbulence, the principal difficulty arises from the loss of information upon averaging, which must in part be restored to arrive at a closed set of equations. A possible way to advance the state of the art is to exploit the great progress in algorithms and computer power to gain enough insight into the small-scale physics of the inter-phase interaction to permit the formulation of better closure models. This approach has led to significant advances e.g. in the understanding of the rheological properties of uniform suspensions in liquids and gases (see e.g. Tsao and Koch, 1995; Sangani et al., 1996; Kang et al., 1997; Brady and Morris, 1997; Ladd, 1997; Morris and Brady, 1998; Koch and Sangani, 1999; Foss and Brady, 2000; Hill et al., 2001 and many others). Much of this progress, however, concerns specific flow situations: simple shear, uniform settling, channel flow, and so on. The next natural step is to attempt a more general formulation, in which a set of equations of relatively broad applicability is developed.

In attempting to reach such a goal one is faced with several difficulties: (i) the need for sufficiently powerful algorithms; (ii) the need to phrase the closure problem in terms of well-defined, computable quantities; (iii) a *systematic procedure* by which the computational results can be used to effect the desired closure. In this paper we address the second and, in part, the third one of these problems for disperse particle–fluid systems. The particles are taken to be equal rigid spheres but, in developing the general theory, no assumptions are made about the Reynolds number of their relative motion with respect to the fluid, nor about the rheological nature of the fluid other than incompressibility. While at sufficiently high particle and global Reynolds numbers the flow will be turbulent, we do not attempt to carry out a Reynolds average. The way in which, for example, large-eddy simulations are built on the laminar Navier–Stokes equations suggests that an effort to derive corresponding "laminar" averaged equations for disperse particle flow is well motivated.

One of the mathematical pathologies exhibited by most of the existing models is their failure at short spatial scales. Physically, the shortest spatial scale which must be relied upon to regularize small-scale behavior must be the finite size of the suspended particles. This consideration suggests that it may be useful to focus on situations in which a competition between the macroscopic scale, L, and the particle radius, a, appears explicitly. Since, in the averaged equations, the former can only arise through the presence of spatial gradients, we are directed to a consideration of spatially non-uniform flows. In this connection, it may be noted that the literature contains ample evidence for the importance of spatial gradients: shear-induced diffusion (Leighton and Acrivos, 1987; Acrivos, 1995; Zarraga and Leighton, 2002), particle migration in pressure-driven flow (Nott and Brady, 1994; Morris and Brady, 1998), band formation in rotating suspensions (Tirumkudulu et al., 1999, 2000), density stratification in sedimentation (Segré et al., 2001; Mucha et al., 2004), volume fraction waves in fluidized beds (Jackson, 2000, Chapter 5), and many others.

The specific objective of this paper is the study of the stress in an incompressible fluid–particle system. As in the classic paper by Batchelor (1970) on the same subject, we use ensemble averaging but, unlike him, we allow for the presence of spatial non-uniformities over length scales L large, but finite, compared with the particle radius a. We find that, when terms of order $(a/L)^2$ are retained in the equations, the stress acquires a non-symmetric contribution even in the absence of couples acting on the particles.

An important aspect of the total stress is the so-called *particle pressure*, a notion introduced by Anderson and Jackson (1967) as "representing the elastic resistance of the [fluidized] particle assembly to compression". It is widely believed that a correct representation of this entity holds the key to solving "one of the simplest and yet most stringent challenges in the modelling of gas-particulate flows, [namely] to predict the conditions under which a homogeneous fluidized bed will be unstable to volume fraction variations" (Koch and Sangani, 1999). Indeed, in another well-known paper by Batchelor (1988) on the subject, for example, it is a particle pressure related to the diffusive transport of particles down a concentration gradient which produces a region of stable fluidization. More recently, the same concept has been used in numerical studies of fluidized bed stability (Göz, 1995; Glasser et al., 1996, 1997; Koch and Sangani, 1999; Sundaresan, 2003), and in the numerical simulation of industrial-scale fluidized beds (see e.g. Gidaspow, 1994). While in a gas–particle system much of the particle pressure would arise from inter-particle collisions, in a liquid–particle system hydrodynamic interactions may constitute the dominant contribution. This is the aspect on which we mostly focus in this paper.

Models also exist in the literature in which the disperse-phase averaged momentum equation does not contain a pressure term (Gidaspow, 1994; Pape and Gidaspow, 1998; Tong and Wang, 1999; Slater and Young, 2001). While this approximation may be justified for small heavy particles, its general validity is questionable (in spite of the attractive feature of leading to hyperbolic equations—see e.g. Lyczkowski et al., 1978). Our conclusion is that a pressure term must be included in the particle momentum equation.

We include a summary of the main results of the paper in the following section, leaving their derivation and generalization for the subsequent sections, with details given in Appendices A–D. For the purpose of illustration of the general results, in Section 6 we consider the specific case of a dilute Stokes-flow suspension for which explicit results are given.

2. Summary of results

Since the details are rather technical, it is useful to start by providing a summary of the main results of this paper, the derivation of which is provided in the following sections and in Appendices A–D.

We use ensemble averages and denote the continuous phase by the subscript 'C' and the disperse phase by the subscript 'D'. Angle brackets will denote phase averages. For example, $\langle \mathbf{u}_C \rangle(\mathbf{x}, t)$ is the phase-average velocity of the continuous phase, i.e. the ensemble average of $\mathbf{u}_C$ taken over all those realizations of the ensemble such that the point $\mathbf{x}$ is in the continuous phase at time t; a formal definition is given in (A.1).

A different type of average, denoted by an over-bar and formally defined in (A.7), also naturally arises in the analysis. For example, $\overline{\mathbf{w}}(\mathbf{x}, t)$ is the value of the particle center-of-mass velocity averaged over all the realization such that a particle is centered at $\mathbf{x}$ at time t. Other particle-average quantities encountered in the following are the average center-of-mass acceleration, the average total hydrodynamic force, and others.

While phase averages describe the local instantaneous fields at $\mathbf{x}$, particle averages describe properties pertaining to the entire particle centered at $\mathbf{x}$. Therefore, due to the finite particle size, these latter averages are in a sense non-local. The two types of average coincide in the case of a spatially uniform suspension, but differ by terms of order (a/L) or $(a/L)^2$ and higher in the presence of spatial gradients. For example, as shown in Appendix A, the disperse-phase volume fraction β_D is related to the number density n by

$$\beta_D(\mathbf{x},t) = \int_{|\mathbf{r}|\leqslant a} d^3r\, n(\mathbf{x}+\mathbf{r},t) \simeq \left(1+\frac{a^2}{10}\nabla^2+\cdots\right)(nv), \tag{1}$$

where $v=\frac{4}{3}\pi a^3$ is the particle volume. As another example, the average velocity $\langle \mathbf{u}_D \rangle$ of the *particle material* is related to the average velocity $\mathbf{w}$ of the *particle center of mass* by

$$\beta_D\langle \mathbf{u}_D \rangle = \left(1+\frac{a^2}{10}\nabla^2+\cdots\right)(nv\overline{\mathbf{w}}) - \frac{1}{5}a^2\nabla\times(nv\overline{\mathbf{\Omega}})+\cdots, \tag{2}$$

where $\overline{\mathbf{\Omega}}$ is the average particle angular velocity, and so forth.

The momentum equation for the continuous phase is found to have a standard form, namely

$$\beta_C\rho_C\langle \mathbf{a}_C \rangle = -\beta_C\nabla p_m + \beta_C\nabla\cdot\mathbf{\Sigma}_m - \left(1+\frac{a^2}{10}\nabla^2\right)(nv\mathbf{F}) + \beta_C\rho_C\mathbf{g}. \tag{3}$$

where β_C is the disperse-phase volume fraction, ρ_C the (microscopic) density, $\langle \mathbf{a}_C \rangle$ the phase-average acceleration, p_m the mixture pressure defined in (17), and $\mathbf{F}$ the phase interaction force defined in (11).

The mixture viscous stress tensor $\mathbf{\Sigma}_m$ differs from standard expressions in that, in general, it has an antisymmetric component even when there are no couples acting on the particles. For a Newtonian continuous phase with viscosity μ_C, $\mathbf{\Sigma}_m$ is found to have the form

$$\mathbf{\Sigma}_m = \mu_C(\nabla\mathbf{u}_m + \nabla\mathbf{u}_m^T) + \mathsf{S} - \boldsymbol{\epsilon}\cdot(\mathbf{A} - \nabla\times\mathbf{B}) \tag{4}$$

in which the superscript 'T' denotes the transpose and $\mathbf{u}_m$ is the volumetric flux:

$$\mathbf{u}_m = \beta_C\langle \mathbf{u}_C \rangle + \beta_D\langle \mathbf{u}_D \rangle. \tag{5}$$

In (4), S is a symmetric traceless two-tensor, $\boldsymbol{\epsilon}$ is the alternating tensor, $\mathbf{A}$ is an axial vector, and $\mathbf{B}$ is a polar vector. The two terms multiplying the alternating tensor constitute the antisymmetric part of the mixture stress, the existence of which is one of the results of this study.

As found by Batchelor (1970), to leading order S is given by the average stresslet:

$$\mathsf{S}_{ij} = n\overline{\int_{|\mathbf{r}|=a} dS_r\left[\frac{1}{2}\Big((\boldsymbol{\sigma}_C\cdot\mathbf{n})_i r_j + (\boldsymbol{\sigma}_C\cdot\mathbf{n})_j r_i\Big) - \frac{a}{3}\delta_{ij}(\mathbf{n}\cdot\boldsymbol{\sigma}_C\cdot\mathbf{n})\right]} + O(a/L), \tag{6}$$

where $\mathbf{n}$ is the unit normal directed outward from the particle. It should be stressed that this result is not limited to low-Reynolds-number flow. The complete expression of S contains additional terms of progressively higher order in a/L, as will be shown in Section 4. It is worth noting that (6) and all the similar relations given later express the components of the average stress in terms of quantities that are explicitly and unambiguously computable by numerical simulation. The present formulation, coupled with extensive numerical simulations, may therefore be of assistance in the closure of the equations. We have undertaken such a project for particles in Stokes flow and some results are given in Marchioro et al. (2000, 2001), Wang and Prosperetti (2001) and Ichiki and Prosperetti (2004a,b).

The leading-order contribution to the axial component of the antisymmetric stress is

$$\mathbf{A} = \frac{1}{2} n \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r \, \mathbf{r} \times (\boldsymbol{\sigma}_{\mathrm{C}} \cdot \mathbf{n})} + \mathrm{O}(a/L) \tag{7}$$

i.e., one half the average hydrodynamic couple on the particle. In general, this term will be non-zero in the presence of angular acceleration of the particles or of external couples acting on them. Again, this result is in agreement with Batchelor (1970). The leading-order contribution to the new polar component of the antisymmetric stress may be written as

$$\mathbf{B} = \frac{1}{10} n a^2 \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r (\mathbf{I} - \mathbf{nn}) \cdot (\boldsymbol{\sigma}_{\mathrm{C}} \cdot \mathbf{n})}, \tag{8}$$

in which **I** is the identity two-tensor. This expression shows that **B** is proportional to the average of the tangential component of the surface traction. When the Reynolds number is not small, the hydrodynamic force is mostly due to the normal component of the stress and therefore this term may be small compared with the total fluid force on the particle. At lower Reynolds number, however, the normal and tangential contributions are comparable. An alternative expression for **B** is

$$\mathbf{B} = -\frac{1}{10} n \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r \, \mathbf{r} \times [\mathbf{r} \times (\boldsymbol{\sigma}_{\mathrm{C}} \cdot \mathbf{n})]}. \tag{9}$$

Either form identifies this term as a force acting on the fluid due to an imbalance of the tangential traction over the particle surface. In the presence of such an imbalance, there will be a net force on the particle and, therefore, on the fluid. As in the case of S, the complete results for both **A** and **B** contain additional terms of progressively higher order in a/L.

Here we only consider equal particles which can be approximated as rigid spheres of radius a. The average equation for their translational motion is found to have the form

$$v \rho_{\mathrm{D}} \overline{\dot{\mathbf{w}}} = v(-\nabla p_{\mathrm{m}} + \nabla \cdot \boldsymbol{\Sigma}_{\mathrm{m}}) + \mathbf{F} + \rho_{\mathrm{D}} v \mathbf{g}, \tag{10}$$

where

$$\mathbf{F} = \mathbf{T} - v(-\nabla p_{\mathrm{m}} + \nabla \cdot \boldsymbol{\Sigma}_{\mathrm{m}}). \tag{11}$$

The term **T** is the average total hydrodynamic force

$$\mathbf{T}(\mathbf{x}, t) = \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r \, \mathbf{n} \cdot \boldsymbol{\sigma}_{\mathrm{C}}(\mathbf{x} + \mathbf{r}, t)}, \tag{12}$$

from which the second term removes the contribution due to the large-scale structure of the flow responsible, among other effects, for the buoyancy force. Thus, **F** may properly be identified with the average fluid force on the particles due to the *local* flow conditions. The collision stress has been neglected in (10) and is introduced later. The advantage of phrasing the disperse-phase momentum equation in terms of the average acceleration of the particle center-of-mass $\overline{\dot{\mathbf{w}}}$, rather than the acceleration $\mathbf{a}_{\mathrm{D}}$ of the particle material, is that in this way the stress internal to the particle need not appear explicitly in the theory. In particular, the rigidity constraint for solid particles can trivially be accounted for.

As a minimum, the two momentum equations (3) and (10) should be complemented by the conservation of the continuous-phase volumetric flux:

$$\frac{\partial \beta_C}{\partial t} + \nabla \cdot \, (\beta_C \langle \mathbf{u}_C \rangle) = 0, \tag{13}$$

by the equation of conservation of the particle number:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\overline{\mathbf{w}}) = 0, \tag{14}$$

and by an equation for the angular momentum of the particles:

$$\frac{2}{5} a^5 \rho_D \overline{\dot{\mathbf{\Omega}}} = \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r \, \mathbf{r} \times (\boldsymbol{\sigma}_C \cdot \mathbf{n})} + \mathbf{L} \tag{15}$$

where $\overline{\dot{\mathbf{\Omega}}}$ is the average angular acceleration and $\mathbf{L}$ the non-hydrodynamic couple. An equation similar to (13) is also satisfied by the phase-average disperse-phase velocity $\langle \mathbf{u}_D \rangle$; adding the two one finds the condition of incompressibility for the total volumetric flux, or mixture velocity $\mathbf{u}_m$:

$$\nabla \cdot \mathbf{u}_m = 0. \tag{16}$$

The mixture pressure p_m is to be found from a solution of the averaged equations system and therefore, in principle, there is no need to express it in terms of the averaged microscopic fields. Nevertheless, the analysis of Sections 4 and 5 shows that, again to leading order,

$$p_m = \beta_C \langle p_C \rangle + \beta_D \left(\frac{1}{4\pi a^2} \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r \, p_C(\mathbf{x} + \mathbf{r}, t)} \right) + O(a/L) \tag{17}$$

in which p_C is the continuous-phase pressure. This result was derived earlier by other means (Marchioro et al., 1999), and a discussion can be found in that reference. The term multiplied by β_D is just the average of the fluid pressure over the surface of the particle centered at $\mathbf{x}$. This quantity arises in other contexts such as, for instance, the theory of the osmotic pressure in a suspension (Brady, 1993; Jeffrey et al., 1993).

Eqs. (3), (10), (13), (14), and (15) have the general form of the so-called "two-fluid model", although they differ in important details from most of the existing formulations. In particular, care has been exercised to retain all terms of order $(a/L)^2$ (not all of which, however, are displayed in the simplified equations presented in this section), in the belief that, if properly modelled, they will regularize the small-scale behavior of the equations which, as is well known, is a matter of serious concern in the formulation of averaged equations models for multiphase flow. Furthermore, the presence of an antisymmetric part of the stress has been explicitly identified. The physical content of the terms introduced in this section will be illustrated in Section 6 by considering the specific example of a dilute suspension in Stokes flow.

3. The total stress

We write the microscopic momentum equation for the continuous (subscript C) and disperse (subscript D) phases in the form

$$\rho_{C,D} \mathbf{a}_{C,D} = \nabla \cdot \boldsymbol{\sigma}_{C,D} - \nabla \psi_{C,D} + \mathbf{b}_{C,D}, \tag{18}$$

where ρ, $\mathbf{a}$, and $\boldsymbol{\sigma}$ denote density, acceleration, and stress tensor; ψ is the potential of the body force (e.g. gravity, in which case $\psi = -\rho \mathbf{x} \cdot \mathbf{g}$), and $\mathbf{b}$ denotes other, non-conservative forces such as those due to particle collisions.

After taking the phase-ensemble average of the two equations according to (A.1) and adding, one finds

$$\beta_C\rho_C\langle \mathbf{a}_C\rangle + \beta_D\rho_D\langle \mathbf{a}_D\rangle = \beta_C\langle \nabla\cdot\boldsymbol{\sigma}_C\rangle + \beta_D\langle \nabla\cdot\boldsymbol{\sigma}_D\rangle - \beta_C\nabla\psi_C - \beta_D\nabla\psi_D + \beta_C\langle \mathbf{b}_C\rangle + \beta_D\langle \mathbf{b}_D\rangle, \tag{19}$$

where we have used the fact that the conservative forces are deterministic. By using the continuity of the normal stresses at the fluid–particle interface and the explicit expression (A.3) of the characteristic function, it is readily established that

$$\beta_C\langle \nabla\cdot\boldsymbol{\sigma}_C\rangle + \beta_D\langle \nabla\cdot\boldsymbol{\sigma}_D\rangle = \nabla\cdot(\beta_C\langle\boldsymbol{\sigma}_C\rangle + \beta_D\langle\boldsymbol{\sigma}_D\rangle), \tag{20}$$

which enables us to introduce the total mixture stress

$$\boldsymbol{\Sigma}_T = \beta_C\langle\boldsymbol{\sigma}_C\rangle + \beta_D\langle\boldsymbol{\sigma}_D\rangle. \tag{21}$$

It is intuitively clear (and can be readily proven from the expression (A.3) of the characteristic function) that the average stress $\langle\boldsymbol{\sigma}_D\rangle$ at a point $\mathbf{x}$ inside the particle phase may be written as (see Eq. (A.6))

$$\beta_D(\mathbf{x})\langle\boldsymbol{\sigma}_D\rangle(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|\leqslant a} \mathrm{d}^3y\, n(\mathbf{y})\langle\boldsymbol{\sigma}_D\rangle_1(\mathbf{x}|\mathbf{y}), \tag{22}$$

where $\langle\boldsymbol{\sigma}_D\rangle_1(\mathbf{x}|\mathbf{y})$ is the average stress at $\mathbf{x}$ conditional on the presence of a particle centered at $\mathbf{y}$; here and in the following we omit the explicit indication of the time variable, which is immaterial for most of the present analysis. The integrand in (22) is a smooth function of the position $\mathbf{y}$ of the particle center, with respect to which it varies over the macroscopic scale L. We can therefore carry out a Taylor series expansion to find (see (A.13))

$$\beta_D\langle\boldsymbol{\sigma}_D\rangle = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\nabla_x^{(k)}\odot\left[n(\mathbf{x})\overline{\int_{r\leqslant a}\mathrm{d}^3r\,(\mathbf{r})^{(k)}\boldsymbol{\sigma}_D}\right], \tag{23}$$

where the notation $\nabla_x^{(k)}\odot$ signifies the kth order divergence with respect to the tensorial indices of the polyadic $\mathbf{r}^{(k)} \equiv \mathbf{rr}\ldots\mathbf{r}$.

It is shown in Appendix B that, by using the condition of continuity of the normal stress at the particle surface and discarding a divergenceless tensor which does not contribute to the momentum equation, Eq. (23) may equivalently be written as

$$\beta_D\langle\boldsymbol{\sigma}_D\rangle = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\nabla_x^{(k)}\odot\left[n(\mathbf{x})\left(\overline{\int \mathrm{d}S_r(\mathbf{r})^{(k)}(\boldsymbol{\sigma}_C\cdot\mathbf{n})\mathbf{r}} - \overline{\int \mathrm{d}^3r\,(\mathbf{r})^{(k)}(\nabla_r\cdot\boldsymbol{\sigma}_D)\mathbf{r}}\right)\right]. \tag{24}$$

For a spatially uniform system, only the first term in the summation would survive giving

$$\beta_D\langle\boldsymbol{\sigma}_D\rangle = n(\mathbf{x})\left[\overline{\int \mathrm{d}S_r(\boldsymbol{\sigma}_C\cdot\mathbf{n})\mathbf{r}} - \overline{\int \mathrm{d}^3r\,(\nabla_r\cdot\boldsymbol{\sigma}_D)\,\mathbf{r}}\right], \tag{25}$$

in agreement with Eq. (4.3) of Batchelor (1970). Furthermore, it is also shown in Appendix B that

$$\beta_D\rho_D\langle \mathbf{a}_D\rangle - \nabla\cdot\left[\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}\nabla_x^{(k)}\cdot\left(n(\mathbf{x})\overline{\int d^3r\,(\mathbf{r})^{(k)}\,(\nabla_r\cdot\boldsymbol{\sigma}_D)\,\mathbf{r}}\right)\right]$$
$$= (nv-\beta_D)\nabla\psi_D + \rho_D n\bar{\dot{\mathbf{w}}} + \beta_D\langle \mathbf{b}_D\rangle - n(\mathbf{x})\overline{\int d^3r\,\mathbf{b}_D(\mathbf{x}+\mathbf{r})}. \tag{26}$$

Thus, the combined momentum equation for the two phases (19) becomes

$$\beta_C\rho_C\langle \mathbf{a}_C\rangle + \rho_D nv\bar{\dot{\mathbf{w}}} = \nabla\cdot\left(\beta_C\langle\boldsymbol{\sigma}_C\rangle + \boldsymbol{\Sigma}^P\right) - \beta_C\nabla\psi_C - nv\nabla\psi_D + n\overline{\int d^3r\,\mathbf{b}_D} + \beta_C\langle \mathbf{b}_C\rangle, \tag{27}$$

where

$$\boldsymbol{\Sigma}^P = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\nabla_x^{(k)}\odot\left[n(\mathbf{x})\left(\overline{\int_{r=a} dS_r\,(\mathbf{r})^{(k)}\,(\boldsymbol{\sigma}_C\cdot\mathbf{n})\,\mathbf{r}}\right)\right], \tag{28}$$

is the particle contribution to the total mixture stress (21). Explicitly, the first few terms are

$$\Sigma_{ij}^P = n\overline{\int dS_r(\boldsymbol{\sigma}_C\cdot\mathbf{n})_i r_j} - \frac{1}{2}\partial_k\left[n\overline{\int dS_r(\boldsymbol{\sigma}_C\cdot\mathbf{n})_i r_j r_k}\right] + \frac{1}{3!}\partial_k\partial_\ell\left[n\overline{\int dS_r(\boldsymbol{\sigma}_C\cdot\mathbf{n})_i r_j r_k r_\ell}\right] + \cdots \tag{29}$$

This quantity will now be manipulated to give the form (4) of the particle stress. The isotropic part (i.e., the part proportional to δ_{ij}) will be identified with the particle pressure.

4. Decomposition of the stress

The expressions (28) and (29) for the particle stress contain tensors of the form

$$T_{ik_1k_2\ldots k_{N-2}k_{N-1}}(\mathbf{x}) = \overline{\int dS_r(\boldsymbol{\sigma}_C\cdot\mathbf{n})_i r_{k_1} r_{k_2}\cdots r_{k_{N-2}} r_{k_{N-1}}} \tag{30}$$

which are symmetric in the indices $k_1, k_2, \ldots, k_{N-1}$. As functions of their indices, each one of these tensors is a reducible representations of the rotation group SO(3), [2] and can be decomposed into a sum of irreducible representations, which constitute the starting point of our analysis; details are provided in Appendix C. A similar remark holds for another class of tensors which arises in the analysis, namely

$$U_{ij\ldots k} \equiv T_{\ell\ell ij\ldots k} = a\overline{\int dS_r(\mathbf{n}\cdot\boldsymbol{\sigma}_C\cdot\mathbf{n})r_i\cdots r_k}. \tag{31}$$

These tensors are evidently totally symmetric in their indices. We denote by U_0 the lowest-order member of this class:

[2] The statement also holds when the $\mathbf{x}$-dependence is considered; from this point of view the T's constitute a reducible representation of SO(3) of much higher order, see Arad et al. (1999).

$$U_0 = a\overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r\, \mathbf{n}\cdot\boldsymbol{\sigma}_\mathrm{C}\cdot\mathbf{n}}. \tag{32}$$

It is shown in Appendix D that, at least for a Newtonian fluid, the viscous part of the stress does not contribute to $\mathbf{n}\cdot\boldsymbol{\sigma}_\mathrm{C}\cdot\mathbf{n}$ so that (31) can also be written as

$$U_{ij\ldots k} = -a\overline{\int \mathrm{d}S_r\, p_\mathrm{C} r_i\cdots r_k}. \tag{33}$$

We can now start to examine the individual terms in the definition (29) of $\boldsymbol{\Sigma}^\mathrm{P}_{ij}$ according to the procedure outlined in Appendix C.

The first term is the second order tensor T_{ij} which can be decomposed as

$$T_{ij} = \widehat{T}_{ij} + \frac{1}{3}\delta_{ij}U_0 + \frac{1}{2}\,(T_{ij} - T_{ji}), \tag{34}$$

where the symmetric traceless component $\widehat{T}_{ij}$ is given by

$$\widehat{T}_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}\delta_{ij}U_0 \tag{35}$$

and coincides with the average stresslet (6). Since U_0 in (34) gives an isotropic contribution to the stress, it will give rise to the first term of the particle pressure. The antisymmetric part of (34) is

$$T^\mathrm{A}_{ij} \equiv \frac{1}{2}\left(T_{ij} - T_{ji}\right) = -\frac{1}{2}\epsilon_{ijp}\overline{\int \mathrm{d}S_r(\mathbf{r}\times(\boldsymbol{\sigma}_\mathrm{C}\cdot\mathbf{n}))_p}, \tag{36}$$

i.e., proportional to the hydrodynamic couple acting on the particle.

Proceeding with the decomposition of the second term of (29) according to the procedure described in Appendix C, we find

$$T_{ijk} = \widehat{T}_{ijk} + \frac{1}{15}[\delta_{ij}(T_k + 2U_k) + \delta_{jk}(T_i + 2U_i) + \delta_{ki}\left(T_j + 2U_j\right)] + T^\mathrm{A}_{ijk} + T^\mathrm{A}_{ikj}, \tag{37}$$

where

$$T^\mathrm{A}_{ijk} = \frac{1}{3}(T_{ijk} - T_{jik}), \quad T^\mathrm{A}_{ikj} = \frac{1}{3}(T_{ijk} - T_{kij}) = \frac{1}{3}(T_{ikj} - T_{kij}). \tag{38}$$

An explicit expression for the traceless symmetric irreducible component $\widehat{T}_{ijk}$ similar to (6) can be readily written down from these definitions. What enters the final momentum equation is the double divergence of (nT_{ijk}) over the indices j and k. Therefore, any tensor Δ_{ijk} such that $\partial_j\partial_k(n\Delta_{ijk}) = 0$ can be added to (37) with no consequences on the physical content of the equations. An examination of $\partial_j\partial_k T_{ijk}$ suggests that the addition of

$$\Delta_{ijk} = \frac{1}{15}[\delta_{ik}(T_j - 4U_j) - \delta_{ij}(T_k - 4U_k)] + \frac{1}{3}(T_{kij} - T_{jik}) \tag{39}$$

gives rise to the simpler and more symmetric expression

$$T'_{ijk} \equiv T_{ijk} + \Delta_{ijk} = \frac{2}{5}U_k\delta_{ij} + \widehat{T}_{ijk} + 2T^\mathrm{A}_{ijk} + \frac{2}{15}\epsilon_{ijs}\epsilon_{sk\ell}(T_\ell - U_\ell) + \frac{1}{5}\delta_{jk}T_i. \tag{40}$$

While there is no particularly compelling reason for this procedure, the physical interpretation of the result is somewhat easier with (40) than with (37). Indeed, after taking the divergence of this expression with respect to the index k, as indicated in (29), the first term is isotropic and, accordingly, will contribute to the mixture pressure. The next, totally symmetric, term contributes to the symmetric stress, and the two following terms to the antisymmetric stress. The nature of the last term will be clarified later in Section 5. It can be verified that, in the case of Stokes flow, (40) coincides with the corresponding term in Eq. (10.3) of Tanksley and Prosperetti (2001).

At the next order we have

$$T_{ijk\ell} = \widehat{T}_{ijk\ell} + \frac{1}{7}\delta_{ijk\ell mn}\widehat{T}_{(mnaa)} - \frac{1}{15}\delta_{ijk\ell}U_0 + T^{A}_{ijk\ell} + T^{A}_{ikj\ell} + T^{A}_{i\ell jk}. \tag{41}$$

Here $\delta_{ijk\ell} = \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}$ and $\delta_{ijk\ell mn}$ is the similarly defined quantity with six indices. Furthermore, $\widehat{T}_{(mnaa)}$ is the traceless part of T_{mnaa} symmetrized over all its indices and is given by

$$\widehat{T}_{(mnaa)} = \frac{1}{2}\left[\frac{1}{2}(T_{mn} + T_{nm}) + U_{mn}\right] - \frac{1}{3}\delta_{mn}U_0, \tag{42}$$

while

$$T^{A}_{ijk\ell} = \frac{1}{4}(T_{ijk\ell} - T_{jik\ell}). \tag{43}$$

As before, we subtract from the right-hand side of (41)

$$\begin{aligned}\Delta_{ijk\ell} = \frac{1}{28}\Big[&\delta_{ij}(T_{k\ell} + T_{k\ell} - 10U_{k\ell}) + 2\delta_{ik}(4U_{j\ell} - T_{j\ell} - T_{\ell j}) + \delta_{i\ell}(T_{jk} + T_{kj} + 2U_{jk}) \\ &+ 4\delta_{jk}(T_{i\ell} + T_{\ell i} - U_{i\ell}) + \delta_{j\ell}(T_{ik} + T_{ki} + 2U_{ik}) + \delta_{k\ell}(2U_{ij} - 5T_{ij} - 5T_{ji}) \\ &+ \frac{4}{5}(2\delta_{ij}\delta_{k\ell} - \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})U_0\Big] + \frac{1}{4}(T_{kij\ell} + T_{\ell ijk} - 2T_{jik\ell})\end{aligned} \tag{44}$$

the triple divergence of which with respect to the last three indices vanishes. With this step we find

$$T'_{ijk\ell} = \frac{3}{7}\delta_{ij}\left(U_{k\ell} - \frac{2}{15}\delta_{k\ell}U_0\right) + \widehat{T}_{ijk\ell} + 3T^{A}_{ijk\ell} + \frac{3}{14}\epsilon_{ijs}\epsilon_{kms}\left[\frac{1}{2}(T_{m\ell} + T_{\ell m}) - U_{m\ell}\right] + \frac{3}{7}\delta_{k\ell}. \tag{45}$$

After taking the double divergence of this expression with respect to the indices k and ℓ, as indicated in (29), the first term is isotropic and will therefore contribute to the mixture pressure. Again, in the case of Stokes flow, (45) coincides with the corresponding term in Eq. (10.3) of Tanksley and Prosperetti (2001).

When the suspending fluid is Newtonian, we have

$$\boldsymbol{\sigma}_C = -p_C\mathbf{I} + \mu_C(\nabla\mathbf{u}_C + \nabla\mathbf{u}_C^{T}). \tag{46}$$

It is readily shown that, due to the no-slip condition, the phase ensemble average of this quantity gives (Zhang and Prosperetti, 1997)

$$\beta_C\langle\boldsymbol{\sigma}_C\rangle = -\beta_C\langle p_C\rangle\mathbf{I} + \mu_C(\nabla\mathbf{u}_m + \nabla\mathbf{u}_m^{T}). \tag{47}$$

If the first term here is combined with the isotropic terms in (34), (40), and (45), we find the following expression for the mixture pressure:

$$\begin{aligned} p_{\mathrm{m}} &= \beta_{\mathrm{C}}\langle p_{\mathrm{C}}\rangle - \frac{1}{3}nU_0 + \frac{1}{5}\partial_k(nU_k) - \frac{1}{14}\partial_k\partial_l\left[n\left(U_{k\ell} - \frac{2}{15}\delta_{k\ell}U_0\right)\right] \\ &= \beta_{\mathrm{C}}\langle p_{\mathrm{C}}\rangle - \frac{1}{3}\left(1 + \frac{a^2}{35}\nabla^2 + \cdots\right)(nU_0) + \frac{1}{5}\left(1 + \frac{a^2}{14}\nabla^2 + \cdots\right)\partial_k(nU_k) - \frac{1}{14}\partial_k\partial_\ell(nU_{k\ell}). \end{aligned} \tag{48}$$

The remaining terms of (34), (40), and (45) can be combined according to their tensorial nature into the terms shown in (4) to find

$$\beta_{\mathrm{C}}\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle + \beta_{\mathrm{D}}\langle\boldsymbol{\sigma}_{\mathrm{D}}\rangle = -p_{\mathrm{m}}\mathbf{I} + \mu_{\mathrm{C}}\left(\nabla\mathbf{u}_{\mathrm{m}} + \nabla\mathbf{u}_{\mathrm{m}}^{\mathrm{T}}\right) + \mathsf{S} + \boldsymbol{\epsilon}\cdot(\mathbf{A} + \nabla\times\mathbf{B}) - \frac{a^2}{10}\partial_j(nT_i). \tag{49}$$

By this procedure, the symmetric part of the particle stress is found to be

$$\mathsf{S}_{ij} = \left(1 + \frac{a^2}{14}\nabla^2 + \cdots\right)(n\widehat{T}_{ij}) - \frac{1}{2}\partial_k(n\widehat{T}_{ijk}) + \frac{1}{6}\partial_k\partial_\ell\left(n\widehat{T}_{ijk\ell}\right) + \cdots, \tag{50}$$

where, here and in the following, the dots stand for the terms in (29) that have been dropped as contributing corrections beyond the order $(a/L)^2$. The tensors $\widehat{T}$ appearing here are given explicitly by (35), (6) and their generalizations.

The axial component of the antisymmetric part of the particle stress is

$$A_i = -\frac{1}{2}n\epsilon_{ijk}T_{jk} + \frac{1}{6}\partial_j[n(\epsilon_{ik\ell}T_{k\ell j} + \epsilon_{jk\ell}T_{k\ell i})] + \cdots \tag{51}$$

It was noted before in connection with (7) and (36) that the first term of the polar vector $\mathbf{A}$ is proportional to the average hydrodynamic couple acting on the particles. It is evident from their definitions (38) and (43) that the subsequent terms are the higher moments of this couple:

$$\epsilon_{ik\ell}T_{k\ell j} + \epsilon_{jk\ell}T_{k\ell i} = \overline{\int\left\{[(\boldsymbol{\sigma}\cdot\mathbf{n})\times\mathbf{r}]_i r_j + [(\boldsymbol{\sigma}\cdot\mathbf{n})\times\mathbf{r}]_j r_i\right\}\mathrm{d}S} \tag{52}$$

The polar vector of the antisymmetric part of the particle stress is

$$B_i = \frac{a}{10}\left(1 + \frac{a^2}{14}\nabla^2\right)n(T_i - U_i) - \frac{a}{21}\partial_j\left[n\left(\frac{1}{2}(T_{ij} + T_{ji}) - U_{ij}\right)\right] + \cdots \tag{53}$$

It was already noted in connection with (8) that the first term is the integral of the tangential viscous stress on the particle surface:

$$T_i - U_i = \overline{a\int_{|\mathbf{r}|=a}\mathrm{d}S_r(\delta_{ij} - n_in_j)\cdot(\boldsymbol{\sigma}_{\mathrm{C}}\cdot\mathbf{n})_j}. \tag{54}$$

From the definitions (30) and (31) of the tensors T and U we find that the second term is related to the first moment of the tangential viscous stress as

$$\frac{1}{2}(T_{ij} + T_{ji}) - U_{ij} = \overline{\frac{a}{2}\int_{r=a}\mathrm{d}S_r\left[(\delta_{i\ell} - n_in_\ell)r_j + \left(\delta_{j\ell} - n_jn_\ell\right)r_i\right]\sigma_{k\ell}n_k}. \tag{55}$$

5. Gauge transformations and the momentum equations

The microscopic momentum equations of both phases, (18), are invariant under a gauge transformation of the form:

$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} + \Phi \mathbf{I}, \quad \psi = \hat{\psi} + \Phi, \tag{56}$$

where Φ is an arbitrary harmonic function. In particular, the continuous-phase pressure transforms according to

$$p_\mathrm{C} = \tilde{p}_\mathrm{C} - \Phi. \tag{57}$$

In spite of its mathematical simplicity, this transformation is physically meaningful: an incompressible material only responds to stress gradients, irrespective of their origin and, therefore, the effect of an external conservative field is equivalent to that of an isotropic stress. Since the mixture we consider here is, as a whole, incompressible, one would expect a property similar to (57) to be enjoyed by the quantity identified with the mixture pressure, in our case p_m given by (48). And, if the mixture pressure does transform as in (57), one would also expect that the viscous part of the particle stress be invariant under the gauge transformation.

In order to verify the covariance of the mixture pressure we use the results, valid for any harmonic function Φ,

$$\int_{r=a} \mathrm{d}S_r\, \Phi(\mathbf{x} + \mathbf{r}) = 4\pi a^2 \Phi(\mathbf{x}), \tag{58}$$

$$a \int_{r=a} \mathrm{d}S_r\, \mathbf{n}\Phi(\mathbf{x} + \mathbf{r}) = v \nabla \Phi(\mathbf{x}), \tag{59}$$

$$a \int_{r=a} \mathrm{d}S_r\, \mathbf{n}\mathbf{n}\Phi(\mathbf{x} + \mathbf{r}) = v \left(\mathbf{I} + \frac{a^2}{5} \nabla\nabla \right) \Phi(\mathbf{x}), \tag{60}$$

to find

$$\begin{aligned} p_\mathrm{m} &= \tilde{p}_\mathrm{m} - \beta_L \Phi - \Phi \left(1 + \frac{a^2}{10} \nabla^2 + \cdots \right) (nv) - \frac{a^4}{70} (\partial_k \partial_\ell nv)(\partial_k \partial_\ell \Phi) + \cdots \\ &= \tilde{p}_\mathrm{m} - \Phi - \frac{a^4}{70} (\partial_k \partial_\ell nv)(\partial_k \partial_\ell \Phi) + \cdots \end{aligned} \tag{61}$$

where we have used (1) to replace $[(1 + (a^2/10)\nabla^2](nv)$ by β_D with an error of order $\mathrm{O}(a^4/L^4)$. The last term contains two more derivatives than were retained in (29) and, therefore, it should be disregarded for consistency. It is likely that it would be cancelled if higher order terms were retained (see Tanksley and Prosperetti, 2001). Thus, we see that the quantity that we identified with the mixture pressure does transform as expected.

To check the invariance of the remainder of the particle stress, we note that all the axial vectors T_{ij}^A etc. contributing to $\mathbf{A}$ are manifestly invariant under the transformation as is evident from their definition, and so is the polar vector $\mathbf{B}$. It only remains to check the symmetric part of the particle stress S. For this purpose we note the results

$$\widehat{T}_{ij} = \widetilde{\widetilde{\widehat{T}}}_{ij} + \frac{a^2}{5} v\partial_i\partial_j\Phi, \tag{62}$$

$$\widehat{T}_{ijk} = \widetilde{\widetilde{\widehat{T}}}_{ijk} + \frac{a^2}{35} v\partial_i\partial_j\partial_k\Phi, \tag{63}$$

$$\widehat{T}_{ijk\ell} = \widetilde{\widetilde{\widehat{T}}}_{ijk\ell} + \frac{a^3}{315} v\partial_i\partial_j\partial_k\partial_\ell\Phi, \tag{64}$$

to find

$$\mathsf{S}_{ij} = \tilde{\mathsf{S}}_{ij} + \frac{a^2}{5}\left[nv\partial_i\partial_j\Phi + \frac{a^2}{14}\partial_k[(\partial_k nv)(\partial_i\partial_j\Phi)] + \cdots\right]. \tag{65}$$

It would be possible to redefine S so as to remove the low-order term $nv\partial_i\partial_j\Phi$. Upon taking the divergence of S, one would then be left with a non-invariant term of order $(a/L)^3$, which can be neglected to the level of approximation retained in this paper. This procedure was followed in Marchioro et al. (1999), where S was redefined by subtracting a proper multiple of $\partial_j T_i + \partial_i T_j$. We do not do this here for reasons which will be explained at the end of this section.

The average momentum equation for the translational motion of the particles is

$$v\rho_D\overline{\dot{\mathbf{w}}} = \mathbf{T} - v\nabla\psi_D + \overline{\int d^3r\,\mathbf{b}_D}, \tag{66}$$

where, as before, ψ_D is the potential of an external force field. In a general flow situation, the average hydrodynamic force $\mathbf{T}$ will depend not only on the local fluid–particle interaction, but also on the large-scale structure of the average stress field. For example, a pressure gradient ∇p_m acting on the mixture (e.g., of hydrostatic origin) would contribute a (possibly "virtual") buoyancy $-v\nabla p_m$ to the hydrodynamic force on each particle. A closure relation for $\mathbf{T}$ can be more easily developed if these large-scale effects are separated from the contribution to the hydrodynamic force only dependent on the local state of motion of the mixture. This consideration suggests to define the local fluid–particle interaction $\mathbf{F}$ by

$$\mathbf{F}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) - \int_{|\mathbf{r}|\leqslant a}(-\nabla p_m + \nabla\cdot\mathbf{\Sigma}_m)(\mathbf{x}+\mathbf{r})d^3r \simeq \mathbf{T}(\mathbf{x}) - v(-\nabla p_m + \nabla\cdot\mathbf{\Sigma}_m), \tag{67}$$

so that Eq. (66) becomes

$$v\rho_D\overline{\dot{\mathbf{w}}} = v(-\nabla p_m + \nabla\cdot\mathbf{\Sigma}_m) - v\nabla\psi_D + \mathbf{F} + \overline{\int d^3r\,\mathbf{b}_D}. \tag{68}$$

Aside from the last term, this is the form given earlier in (10). It may be observed that $\mathbf{F}$ defined by (67) is invariant upon a gauge transformation, which indicates that this quantity is insensitive to external pressure gradients applied to the system, as would be expected from a force dependent only on the local flow conditions. A practical and very useful consequence of this fact is that the closure relation for $\mathbf{F}$ defined by (67) cannot contain ∇p_m. As a further justification for the definition (67) of the inter-phase force, one may think of the stress at the particle surface as consisting of two components, one, embodied in the mixture stress, due to the slow spatial variation of the flow, and one arising from the local flow around the particle. If this decomposition is adopted in the calculation of $\mathbf{T}$, a Taylor series expansion of the mixture stress around the particle center would give the last term in (67), while the remainder would be identified with **F**. A similar heuristic argument was used in Prosperetti and Jones (1984).

If now (68) is used to eliminate $\bar{\mathbf{w}}$ from (27), one finds

$$\beta_C \rho_C \langle \mathbf{a}_C \rangle = -\beta_C \nabla p_m + \beta_C \nabla \cdot \mathbf{\Sigma}_m - \beta_C \nabla \psi_C - \left(1 + \frac{a^2}{10} \nabla^2\right)(n\mathbf{F}), \tag{69}$$

where the terms

$$\frac{a^2}{10}\left[2(\nabla n v) \cdot \nabla(-\nabla p_m + \nabla \cdot \mathbf{\Sigma}_m) + n v \nabla^2 \nabla \cdot (-p_m \mathbf{I} + \mathbf{\Sigma}_m)\right]. \tag{70}$$

have been dropped in the right-hand side as being of higher order. An additional justification for this step is that the appearance of derivatives of the pressure higher than the first would alter the mathematical structure of the equations and raise issues such as additional boundary conditions about which virtually nothing is known at present. The last term $(a^2/10)\nabla^2(n\mathbf{F})$ in of (69) arises from the last term of (40). The significance of this term is made clearer by noting that (Tanksley and Prosperetti, 2001)

$$\frac{1}{v}\int_{r \leqslant a} \mathrm{d}^3 r\, n(\mathbf{x}+\mathbf{r})\mathbf{F}(\mathbf{x}+\mathbf{r}) = \left(1 + \frac{a^2}{10}\nabla^2 + \cdots\right)(n\mathbf{F}). \tag{71}$$

It is evident that the integral in the left-hand side is the proper form for the average force per unit volume exerted by the particles on the fluid at point **x**, and the terms in the right-hand side approximate it to order $(a/L)^2$.

If the symmetric stress (65) had been made gauge invariant as mentioned before, the result would have been a different form for the force **F** and a slight change in (70). These alternative expressions are readily derived, but they do not seem to offer any advantage over the ones used here.

6. An example

To first order in the disperse-phase volume fraction β_D, the particles can be assumed to be immersed in the average fields (see e.g. Zhang and Prosperetti, 1997), and it is then easy to calculate the integrals of type T and U arising in the expressions of Section 4 for the various contributions to the stress.

For the case of particles in Stokes flow, the components of the stress are found to be

$$\mathsf{S} = \frac{5}{2}\beta_D \mu_C (\nabla \mathbf{u}_m + \nabla \mathbf{u}_m), \tag{72}$$

$$\mathbf{A} = -3\beta_D \mu_C \left(\overline{\mathbf{\Omega}} - \frac{1}{2}\nabla \times \mathbf{u}_m\right), \tag{73}$$

$$\mathbf{B} = \frac{3}{10}\beta_D \mu_C (\mathbf{u}_m - \overline{\mathbf{w}}). \tag{74}$$

Consistent with the previous analysis, in writing these expressions, we have retained only terms which contribute at most terms of order $(a/L)^2$ in the average equations, as these terms result at most in second-order derivatives of the velocity fields. To the same accuracy, β_D has been substituted for nv whenever the resulting error is smaller than $(a/L)^2$.

Since the particle momentum equation is already of order β_D, the expressions for p_m and $\mathbf{\Sigma}_m$ to be used in it should be reduced to those of the pure fluid so that $-\nabla p_m + \nabla \cdot \mathbf{\Sigma}_m = \nabla \psi_C$. With this simplification, and upon recalling Faxén's theorem, the interphase force (67) is

$$\mathbf{F} = 6\pi\mu_C a\left(\mathbf{u}_m - \overline{\mathbf{w}} + \frac{a^2}{6}\nabla^2\mathbf{u}_m\right), \tag{75}$$

so that, with the neglect of inertia, the particle equation of motion (66) becomes

$$6\pi\mu_C a\left(\mathbf{u}_m - \overline{\mathbf{w}} + \frac{a^2}{6}\nabla^2\mathbf{u}_m\right) - v\nabla(\psi_D - \psi_C) = 0. \tag{76}$$

When the potential ψ is due to gravity, the last term is just the particle weight corrected for buoyancy.

Although, as noted before, there is no need for a constitutive relation for the mixture pressure, which is to be calculated from the average equations themselves, it is interesting to exhibit its form in this case. One finds

$$p_m = \beta_C\langle p_C\rangle + \left(1 + \frac{a^2}{10}\nabla^2\right)(nv\langle p_C\rangle) + \frac{3}{10}\mu_C\nabla\cdot[\beta_D(\mathbf{u}_m - \overline{\mathbf{w}})], \tag{77}$$

or, if only first-order derivatives of the pressure are allowed in the final equations for the reason noted at the end of the previous section,

$$p_m = \langle p_C\rangle + \frac{3}{10}\mu_C\nabla\cdot[\beta_D(\mathbf{u}_m - \overline{\mathbf{w}})]. \tag{78}$$

The particles' size and their relative velocity with respect to the fluid may justify a Stokes-flow approximation. The characteristic length and velocity of the average fluid motion, however, are not necessarily small and it is therefore consistent to retain fluid inertia. With this remark, and with the previous results (72)–(74), the continuous-phase momentum equation is

$$\begin{aligned}\beta_C\rho_C\langle\mathbf{a}_C\rangle = {} & -\beta_C\nabla p_m + \beta_C\nabla\cdot\left[\mu_{\text{eff}}\left(\nabla\mathbf{u}_m + \nabla\mathbf{u}_m^T\right)\right] + 3\nabla\times\left[\beta_D\mu_C\left(\overline{\mathbf{\Omega}} - \frac{1}{2}\nabla\times\mathbf{u}_m\right)\right] \\ & + \frac{3}{10}\nabla\times\nabla\times[\beta_D\mu_C(\mathbf{u}_m - \overline{\mathbf{w}})] - \left(1 + \frac{a^2}{10}\nabla^2\right)(n\mathbf{F}) - \beta_C\nabla\psi_C\end{aligned} \tag{79}$$

in which

$$\mu_{\text{eff}} = \left(1 + \frac{5}{2}\beta_D\right)\mu_C \tag{80}$$

is the well-known result for the effective viscosity to this order. By (75) and (78) for the mixture pressure, (79) may also be written as

$$\beta_C\rho_C\langle \mathbf{a}_C\rangle = -\beta_C\nabla\langle p_C\rangle + \beta_C\nabla\cdot[\mu_{\mathrm{eff}}(\nabla\mathbf{u}_m + \nabla\mathbf{u}_m^T)] + 3\nabla\times\left[\beta_D\mu_C\left(\overline{\mathbf{\Omega}} - \frac{1}{2}\nabla\times\mathbf{u}_m\right)\right] + \frac{3}{4}\nabla^2[\beta_D\mu_C(\overline{\mathbf{w}} - \mathbf{u}_m)] - 6\pi\mu_C na\left(\mathbf{u}_m - \overline{\mathbf{w}} + \frac{a^2}{6}\nabla^2\mathbf{u}_m\right) - \beta_C\nabla\psi_C. \tag{81}$$

To first order in β_D, this equation and (76) coincide with the results given earlier in Zhang and Prosperetti (1997). They differ from the form which could be written down a priori simply by using an effective viscosity and adding the force exerted by one particle on the fluid multiplied by n by the term $\frac{3}{4}\nabla^2[\beta_D\mu_C(\overline{\mathbf{w}} - \mathbf{u}_m)]$ which arises in part from the antisymmetric component of the stress, and in part from the term $a^2\nabla^2/10$ in the parentheses multiplying the force in (79). Both these terms are a consequence of the finite extent of the particles. It should be noted that these contributions have the same order of magnitude as the Faxén contribution to the force, so that it would be inconsistent to disregard them if the latter is retained.

7. Conclusions

The objective of this paper was to present an analysis of the particle contribution to the stress in an incompressible disperse fluid–particle system. We have found that, in general, the particle stress consists of a symmetric and an antisymmetric part. In the simple example given in Section 6, the symmetric part simply contributes to the effective viscosity. At higher volume fractions, it is possible that other contributions would arise as suggested by the results of Marchioro et al. (2001).

The antisymmetric stress consists of two parts, both related to the tangential viscous stresses on the particle surface. The first part is just the mean hydrodynamic couple acting on the particles and its higher moments. This contribution arises due to a relative rotation of the particles with respect to the surrounding fluid and represents a momentum source due to the spatial variation of the angular velocity of this rotational motion.

The second part, which enters the averaged momentum equations at a higher order in the spatial derivatives, is the contribution to the hydrodynamic force on the particle due to an imbalance of the tangential component of the viscous traction at the particle surface. This second component of the antisymmetric stress does not seem to have been identified before.

In the course of the analysis, the contribution of the particle stress to the mixture pressure has also been identified. This contribution coincides with the one derived earlier (Marchioro et al., 1999; Tanksley and Prosperetti, 2001) by different methods.

As a further result of the analysis, a form for the momentum equations of the two phases has been proposed and justified. In the case of dilute Stokes flow the disperse-phase momentum equation takes on the expected form, while the continuous-phase momentum equation acquires additional contributions which were first found in Zhang and Prosperetti (1997) on the basis of a different analysis.

Many of the new terms that we find would vanish if a nearly spatially uniform system had been considered, which explains why they were missed in the past. With an eye toward exploring the effects of spatial non-uniformity, we have purposely carried terms up to second order in the ratio a/L of the particle radius to the macroscopic scale. Our motivation was twofold. In the first place, recent research points out several instances in which spatial non-uniformities seem to be important; examples are shear-induced diffusion, banding, and velocity fluctuations. Secondly, most of the averaged equations models developed so far exhibit unphysical pathologies at the small spatial scales and it may be thought that the situation will be improved by carrying additional terms which depend explicitly on the shortest scale of the system, namely the particle radius.

While we have only shown explicit results for particles in dilute Stokes flow, the results given are general and exhibit the useful feature that the closure problem that needs to be solved to develop a closed set of equations is phrased in terms of computable quantities. Examples of how this circumstance may open the way to a numerically-aided procedure for the development of a systematic closure have been presented in Marchioro et al. (2001), Wang and Prosperetti (2001), and Ichiki and Prosperetti (2004a,b).

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Appendix A. Phase and particle averages

We consider N identical spherical particles, each one with its own degrees of freedom of position, velocity, orientation, and angular velocity, in an incompressible continuous phase. We denote by $\mathscr{C}$ a specific configuration of the system in its phase space, by $\mathscr{P}(\mathscr{C})$ the probability density of this configuration with an associated suitable measure $d\mathscr{C}$, and by $\chi_{C,D}$ the characteristic, or indicator, functions of the two phases. Since the fluid–particle interfaces have measure zero, $\chi_C + \chi_D = 1$. We omit explicit indication of the time variable throughout.

The phase ensemble average of a generic field $f_{C,D}$ is defined by

$$\beta_{C,D}\langle f_{C,D}\rangle(\mathbf{x}) = \frac{1}{N!}\int d\mathscr{C}\,\mathscr{P}(\mathscr{C})\,\chi_{C,D}(\mathbf{x};\mathscr{C})f_{C,D}(\mathbf{x};\mathscr{C}), \tag{A.1}$$

where the division by $N!$ reflects the identity of the particles and $\beta_{C,D}$ are the local instantaneous volume fractions defined by

$$\beta_{C,D}(\mathbf{x}) = \frac{1}{N!}\int d\mathscr{C}\,\mathscr{P}(\mathscr{C})\,\chi_{C,D}(\mathbf{x};\mathscr{C}), \tag{A.2}$$

i.e., the probabilities that, at time t, the position $\mathbf{x}$ is occupied by the C- or D-phase. The probability density is normalized in the usual way so that $\beta_C + \beta_D = 1$.

An explicit representation of the D-phase characteristic function is

$$\chi_D = \sum_{\alpha=1}^{N} H(a - |\mathbf{x} - \mathbf{y}^\alpha|), \tag{A.3}$$

with H the Heaviside distribution and a the particle radius. Upon substitution into (A.1), in view of the identity of the particles, we have

$$\beta_{\mathrm{D}}(\mathbf{x})\langle f_{\mathrm{D}}\rangle(\mathbf{x}) = \frac{1}{(N-1)!}\int_{|\mathbf{x}-\mathbf{y}^1|\leqslant a} \mathrm{d}^3y^1 \int \mathrm{d}\mathscr{C}'\,\mathscr{P}(\mathbf{y}^1,\mathscr{C}')f_{\mathrm{D}}(\mathbf{x};\mathscr{C}), \tag{A.4}$$

where $\mathscr{C}'$ denotes all the state variables of the system with the exclusion of $\mathbf{y}^1$, the position of particle 1. We now write $\mathscr{P}(\mathbf{y}^1,\mathscr{C}) = \mathscr{P}(\mathscr{C}|\mathbf{y}^1)n(\mathbf{y}^1)$, where $\mathscr{P}(\mathscr{C}|\mathbf{y}^1)$ is the conditional probability density and n the particle number density, defined presently in (A.9), and define the conditional average of f_{D} by [3]

$$\langle f_{\mathrm{D}}\rangle_1(\mathbf{x}|\mathbf{y}) = \frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}'\,\mathscr{P}(\mathscr{C}|\mathbf{y})f_{\mathrm{D}}(\mathbf{x};\mathscr{C}). \tag{A.5}$$

Eq. (A.4) then becomes

$$\beta_{\mathrm{D}}(\mathbf{x})\langle f_{\mathrm{D}}\rangle(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|\leqslant a} \mathrm{d}^3y\,n(\mathbf{y})\,\langle f_{\mathrm{D}}\rangle_1(\mathbf{x}|\mathbf{y}) \tag{A.6}$$

which was used in (22).

The field f_{D} is a quantity distributed throughout the volume of each particle. However, there also exist quantities, such as the center-of-mass velocity, angular velocity, and many others, associated with each particle as a whole. For quantities of this type, it is convenient to use a different average, the *particle average*, indicated by an overline and defined, for a generic quantity g^α belonging to particle α, by

$$n(\mathbf{x})\overline{g}(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}\,\mathscr{P}(\mathscr{C})\left[\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{y}^\alpha)g^\alpha(\mathscr{C})\right] \tag{A.7}$$

or, upon introduction of the conditional probability,

$$\overline{g}(\mathbf{x}) = \frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}'\,\mathscr{P}(\mathscr{C}|\mathbf{x})g^1(\mathbf{x},\mathscr{C}'). \tag{A.8}$$

The notation explicitly indicates that the value of g^α for particle α in general depends on the entire configuration of the system. The particle number density n is defined by

$$n(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}\,\mathscr{P}(\mathscr{C})\left[\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{y}^\alpha)\right]. \tag{A.9}$$

In order to prove (23), we observe that, away from boundaries and sharp transition zones, $\langle f_{\mathrm{D}}\rangle_1$ varies slowly with the position $\mathbf{y}$ of the particle center. We exploit this fact by setting $\mathbf{y} = \mathbf{x} - \mathbf{r}$ and defining

$$F(\mathbf{x}-\mathbf{r};\mathbf{s}) \equiv n(\mathbf{x}-\mathbf{r})\langle f_{\mathrm{D}}\rangle_1(\mathbf{s}+\mathbf{x}-\mathbf{r}|\mathbf{x}-\mathbf{r}), \tag{A.10}$$

[3] This is a somewhat simplified definition valid when, as in this application (cf. the domain for the $\mathbf{y}^1$ integration in (A.4)), the point $\mathbf{x}$ is in the particle.

where $\mathbf{s}$ will eventually be taken equal to $\mathbf{r}$. A Taylor series expansion in $\mathbf{r}$ then gives

$$F(\mathbf{x}-\mathbf{r};\mathbf{s}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\mathbf{r})^{(k)} \odot \nabla_x^{(k)} F(\mathbf{x};\mathbf{s}), \tag{A.11}$$

where $(\mathbf{r})^{(0)} \odot \nabla_x^{(0)} = 1$, $(\mathbf{r})^{(1)} \odot \nabla_x^{(1)} = \mathbf{r}\cdot\nabla_x$, $(\mathbf{r})^{(2)} \odot \nabla_x^{(2)} = \mathbf{rr} : \nabla_x\nabla_x$, and so on. Upon substitution into (A.6), and use of the definition (A.8) of particle average, we then find

$$\beta_{\mathrm{D}}\langle f_{\mathrm{D}}\rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \nabla_x^{(k)} \odot \left[n(\mathbf{x}) \overline{\int_{r\leqslant a} \mathrm{d}^3 r\, (\mathbf{r})^{(k)} f_{\mathrm{D}}(\mathbf{x}+\mathbf{r},\mathscr{C}')} \right]. \tag{A.12}$$

It may be noted that, for a spatially uniform system, this reduces to

$$\beta_{\mathrm{D}}\langle f_{\mathrm{D}}\rangle = n(\mathbf{x}) \overline{\int_{r\leqslant a} \mathrm{d}^3 r\, f_{\mathrm{D}}(\mathbf{x}+\mathbf{r},\mathscr{C}')}. \tag{A.13}$$

For example, if f_{D} is the velocity $\mathbf{u}_{\mathrm{D}}$ of a point located at distance $\mathbf{r}$ from the center of a rigid particle translating with velocity $\mathbf{w}$ and rotating with angular velocity $\boldsymbol{\Omega}$, we have $\mathbf{u}_{\mathrm{D}} = \mathbf{w} + \boldsymbol{\Omega}\times\mathbf{r}$ and this expression gives

$$\beta_{\mathrm{D}}\langle \mathbf{u}_{\mathrm{D}}\rangle = nv\overline{\mathbf{w}}, \tag{A.14}$$

as expected. Similarly, upon taking $f_{\mathrm{D}} = 1$, we find $\beta_{\mathrm{D}} = nv$ which this derivation shows to be valid only in the spatially uniform case. With $f_{\mathrm{D}} = 1$, the general expression (A.12) gives

$$\beta_{\mathrm{D}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \nabla_x^{(k)} \odot \left[n(\mathbf{x}) \overline{\int_{r\leqslant a} \mathrm{d}^3 r\, (\mathbf{r})^{(k)}} \right] \tag{A.15}$$

which, truncated to the first three terms, is (1). A similar procedure applied to $\mathbf{u}_{\mathrm{D}} = \mathbf{w} + \boldsymbol{\Omega}\times\mathbf{r}$ gives (2).

Appendix B. Proof of (24) and (26)

Here we start from Eq. (23):

$$\beta_{\mathrm{D}}\langle \boldsymbol{\sigma}_{\mathrm{D}}\rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \nabla_x^{(k)} \odot \left[n(\mathbf{x}) \overline{\int_{r\leqslant a} \mathrm{d}^3 r\, (\mathbf{r})^{(k)} \boldsymbol{\sigma}_{\mathrm{D}}} \right] \tag{B.1}$$

and show how the results (24) and (26) are derived. An important aspect of the proof is that what is dynamically significant is not $\beta_{\mathrm{D}}\langle \boldsymbol{\sigma}_{\mathrm{D}}\rangle$ but its divergence.

The first step is to generalize Batchelor's treatment of the first term of (B.1) (which is the only one that survives in the uniform case) to the general situation. Batchelor (1970) observes that

$$\int \mathrm{d}^3 r\, (\sigma_{\mathrm{D}})_{ij} = \int \mathrm{d}^3 r \left[\partial_j \big(r_k (\sigma_{\mathrm{D}})_{ik} \big] - r_k \partial_j (\sigma_{\mathrm{D}})_i k \right] = a \int \mathrm{d}S\, n_j (\boldsymbol{\sigma}_{\mathrm{C}}\cdot\mathbf{n})_i - \int \mathrm{d}^3 r\, r_k \partial_j (\sigma_{\mathrm{D}})_{ik} \tag{B.2}$$

which is (25) and Eq. (4.3) of Batchelor (1970). To see how to proceed for the higher-order terms, let us consider, for example, the third term of (B.1); for simplicity of writing we drop the subscript 'D':

$$\int d^3r\, r_p r_q \sigma_{ij} = \frac{1}{3}\int d^3r\,[r_p r_q \sigma_{ij} + r_q r_j \sigma_{ip} + r_j r_p \sigma_{iq}] + \frac{1}{3}\int d^3r\,[r_q(r_p\sigma_{ij} - r_j\sigma_{ip}) + r_p(r_q\sigma_{ij} - r_j\sigma_{iq})]. \tag{B.3}$$

The two terms in the second integral are antisymmetric in (p, j) and (q, j), respectively, and therefore they vanish when they are contracted with $\partial_p\partial_q\partial_j$. As for the terms in the first integral, we have

$$\begin{aligned} r_p r_q \sigma_{ij} + r_q r_j \sigma_{ip} + r_j r_p \sigma_{iq} &= [r_p r_q \delta_{jk} + r_q r_j \delta_{pk} + r_j r_p \delta_{qk}]\sigma_{ik} = [\partial_k(r_j r_p r_q)]\sigma_{ik} \\ &= \partial_k(r_j r_p r_q \sigma_{ik}) - r_j r_p r_q \partial_k \sigma_{ik} \end{aligned} \tag{B.4}$$

so that

$$\int d^3r\, r_p r_q \sigma_{ij} = \frac{1}{3}\int dS\, r_j r_p r_q (\boldsymbol{\sigma}_C \cdot \mathbf{n})_i - \frac{1}{3}\int d^3r\, r_j r_p r_q (\nabla \cdot \boldsymbol{\sigma}_D)_i. \tag{B.5}$$

The same procedure generates all the other terms of (24).

To prove (26), we now consider the divergence of the second group of terms generated by the previous procedure, namely the second group of terms in (24):

$$\begin{aligned} \nabla \cdot \mathscr{B} &\equiv \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \nabla_x^{(k+1)} \odot \left[n(\mathbf{x}) \overline{\int_{r\leqslant a} d^3r\, (\mathbf{r})^{(k)} (\nabla_r \cdot \boldsymbol{\sigma}_D)\mathbf{r}} \right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} \nabla_x^{k} \odot \left[n(\mathbf{x}) \overline{\int_{r\leqslant a} d^3r\, (\mathbf{r})^{(k)} (\nabla_r \cdot \boldsymbol{\sigma}_D)} \right] - n(\mathbf{x}) \overline{\int_{r\leqslant a} d^3r\, (\nabla_r \cdot \boldsymbol{\sigma}_D)}. \end{aligned} \tag{B.6}$$

From (A.8) we may write

$$n(\mathbf{x}) \overline{\int d^3r\, (\mathbf{r})^{(k)} (\nabla_r \cdot \boldsymbol{\sigma}_D)} = \frac{1}{(N-1)!} \int d\mathscr{C}'\, \mathscr{P}(\mathbf{x}, \mathscr{C}') \int d^3r\, (\mathbf{r})^{(k)} (\nabla_r \cdot \boldsymbol{\sigma}_D)(\mathbf{x} + \mathbf{r}, \mathscr{C}'). \tag{B.7}$$

We now apply the formula for the derivative of a product finding

$$\begin{aligned} &\nabla^{(k)} \odot \frac{1}{(N-1)!} \int d\mathscr{C}' \mathscr{P}(\mathbf{x}, \mathscr{C}') \int d^3r\, (\mathbf{r})^{(k)} (\nabla_r \cdot \boldsymbol{\sigma}_D) \\ &\quad = \frac{1}{(N-1)!} \sum_{p=0}^{k} \binom{k}{p} \left[\int d\mathscr{C}'\, \underbrace{\nabla\nabla \ldots \nabla}_{p} \mathscr{P}(\mathbf{x}, \mathscr{C}') \right] \int d^3r\, (\mathbf{r})^{(k)} \left[\underbrace{\nabla\nabla \ldots \nabla}_{k-p} (\nabla_r \cdot \boldsymbol{\sigma}_D) \right]. \end{aligned} \tag{B.8}$$

Upon summming over k and inverting the order of summation, we find

$$\begin{aligned} \nabla \cdot \mathscr{B} = &\int d^3r \frac{1}{(N-1)!} \int d\mathscr{C}' \sum_{p=0}^{\infty} \frac{(-\mathbf{r})^{(p)}}{p!} \odot \left[\underbrace{\nabla\nabla \ldots \nabla}_{p} \mathscr{P}(\mathbf{x}, \mathscr{C}') \right] \times \sum_{n=0}^{\infty} \frac{(-\mathbf{r})^{(n)}}{n!} \\ &\odot \left[\underbrace{\nabla\nabla \ldots \nabla}_{n} [(\nabla \cdot \boldsymbol{\sigma}_D)(\mathbf{x} + \mathbf{r}, \mathscr{C}')] \right] - n(\mathbf{x}) \overline{\int d^3r\, (\nabla_r \cdot \boldsymbol{\sigma}_D)}. \end{aligned} \tag{B.9}$$

The two series can be summed with Taylor's theorem so that

$$\nabla\cdot\mathscr{B} = \frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}'[(\nabla\cdot\boldsymbol{\sigma}_{\mathrm{D}})(\mathbf{x},\mathscr{C}')]\int_{r\leqslant a}\mathrm{d}^3r\mathscr{P}(\mathbf{x}-\mathbf{r},\mathscr{C}') - n(\mathbf{x})\overline{\int_{r\leqslant a}\mathrm{d}^3r\,(\nabla_r\cdot\boldsymbol{\sigma}_{\mathrm{D}})}$$
$$= \frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}'[(\nabla\cdot\boldsymbol{\sigma}_{\mathrm{D}})(\mathbf{x},\mathscr{C}')]\int_{r\leqslant a}\mathrm{d}^3r\mathscr{P}(\mathbf{x}+\mathbf{r},\mathscr{C}') - n(\mathbf{x})\overline{\int_{r\leqslant a}\mathrm{d}^3r\,(\nabla_r\cdot\boldsymbol{\sigma}_{\mathrm{D}})}. \tag{B.10}$$

We now substitute for $\nabla\cdot\boldsymbol{\sigma}_{\mathrm{D}}$ its expression from the momentum equation (18) for the particle material:

$$\nabla\cdot\boldsymbol{\sigma}_{\mathrm{D}} = \rho_{\mathrm{D}}\mathbf{a}_{\mathrm{D}} + \nabla\psi_{\mathrm{D}} - \mathbf{b}_{\mathrm{D}}. \tag{B.11}$$

Since ψ_{D} is deterministic, recalling the definition of the disperse-phase volume fraction, we have

$$\frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}'\nabla\psi_{\mathrm{D}}(\mathbf{x})\int \mathrm{d}^3r\mathscr{P}(\mathbf{x}+\mathbf{r},\mathscr{C}') - n(\mathbf{x})\overline{\int \mathrm{d}^3r\,\nabla\psi_{\mathrm{D}}} = (\beta_{\mathrm{D}} - nv)\nabla\psi_{\mathrm{D}}(\mathbf{x}). \tag{B.12}$$

For the acceleration term, from (A.4),

$$\frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}'\mathbf{a}_{\mathrm{D}}(\mathbf{x}|\mathscr{C})\int \mathrm{d}^3r\mathscr{P}(\mathbf{x}+\mathbf{r},\mathscr{C}') - n(\mathbf{x})\overline{\int \mathrm{d}^3r\,\mathbf{a}_{\mathrm{D}}} = \beta_{\mathrm{D}}\langle\mathbf{a}_{\mathrm{D}}\rangle - nv\bar{\dot{\mathbf{w}}} \tag{B.13}$$

and, similarly,

$$\frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}'\mathbf{b}_{\mathrm{D}}(\mathbf{x}|\mathscr{C})\int \mathrm{d}^3r\mathscr{P}(\mathbf{x}+\mathbf{r},\mathscr{C}') - n(\mathbf{x})\overline{\int \mathrm{d}^3r\,\mathbf{b}_{\mathrm{D}}\mathbf{r}} = \beta_{\mathrm{D}}\langle\mathbf{b}_{\mathrm{D}}\rangle - n\overline{\int \mathrm{d}^3r\,\mathbf{b}_{\mathrm{D}}}. \tag{B.14}$$

With these results, we find (26).

Appendix C. Decomposition of the stress

The decomposition of the stress adopted in Section 4 rests on the interpretation of the tensors **T** and **U** as reducible representations of the rotation group. Here we follow the work of Damour and Iyer (1991), whose notation we also adopt.

If $T_{ij\ldots rs}$ is an N-tensor, we denote by $T_{(ij\ldots rs)}$ the tensor obtained by a total symmetrization of its indices. Clearly

$$T_{(ij\ldots rs)} = \frac{1}{N}\left[T_{i(j\ldots rs)} + T_{j(i\ldots rs)} + \cdots + T_{r(ij\ldots s)} + T_{s(ij\ldots r)}\right] \tag{C.1}$$

or, since in our case the **T**'s are symmetric in all their indices except the first one,

$$T_{(ij\ldots rs)} = \frac{1}{N}\left(T_{ij\ldots rs} + T_{ji\ldots rs} + \cdots + T_{rij\ldots s} + T_{sij\ldots r}\right). \tag{C.2}$$

With this result we may write

$$T_{ij\dots rs} = T_{(ij\dots rs)} + \frac{N-1}{N} T_{ij\dots rs} - \frac{1}{N}(T_{ji\dots rs} + \cdots + T_{rij\dots s} + T_{sij\dots r})$$
$$= T_{(ij\dots rs)} + \frac{1}{N}[(T_{ij\dots rs} - T_{ji\dots rs}) + \cdots + (T_{ij\dots rs} - T_{rij\dots s}) + (T_{ij\dots rs} - T_{sij\dots r})], \tag{C.3}$$

where the brackets contain $N-1$ pairs of terms.

The symmetric traceless part $\widehat{T}$ of T is the only irreducible representation of order N of SO(3), the rotation group in three dimensions, and is given by (Damour and Iyer, 1991)

$$\widehat{T}_{ijk_1k_2\dots k_{N-2}} = T_{(ijk_1k_2\dots k_{N-2})} + \sum_{k=1}^{[N/2]} a_k^N \delta_{(i_1i_2\dots}\delta_{i_{2k-1}i_{2k}} S_{i_{2k+1}\dots i_N)a_1a_1\dots a_ka_k}, \tag{C.4}$$

where $[N/2]$ denotes the integer part of $N/2$ and the parentheses enclosing the indices indicate a total symmetrization; the convention of summation over repeated indices is adhered to. The first few coefficients are given by

$$a_0^N = 1, \quad a_1^N = -\frac{1}{2}\frac{N(N-1)}{2N-1}, \quad a_2^N = \frac{1}{8}\frac{N(N-1)(N-2)(N-3)}{(2N-1)(2N-3)}. \tag{C.5}$$

With (C.3) and (C.4), T may be decomposed as

$$T_{ij\dots rs} = \widehat{T}_{ij\dots rs} - \sum_{k=1}^{[N/2]} a_k^N \delta_{(i_1i_2\dots}\delta_{i_{2k-1}i_{2k}} S_{i_{2k+1}\dots i_N)a_1a_1\dots a_ka_k}$$
$$+ \frac{1}{N}[(T_{ij\dots rs} - T_{ji\dots rs}) + \cdots + (T_{ij\dots rs} - T_{rij\dots s}) + (T_{ij\dots rs} - T_{sij\dots r})]. \tag{C.6}$$

Upon noting that $T_{\ell\ell} = U_0$, application of this relation to the second-order tensor T_{ij} gives directly (34), where it will be recognized that the first term is an irreducible representation of order 2, the second term an irreducible representations of order 0, and the last term an irreducible representations of order 1.

At the next order, we find

$$\sum_{k=1}^{1} a_k^3 \delta_{(i_1i_2}S_{i_3)aa} = -\frac{3}{5}\frac{1}{3}\left[\delta_{ij}S_{kaa} + \delta_{jk}S_{iaa} + \delta_{jk}S_{iaa}\right] = -\frac{1}{5}\left[\delta_{ij}T_{(kaa)} + \delta_{jk}T_{(iaa)} + \delta_{ki}T_{(jaa)}\right] \tag{C.7}$$

and, therefore, (C.6) gives

$$T_{ijk} = \widehat{T}_{ijk} + \frac{1}{5}[\delta_{ij}T_{(kaa)} + \delta_{jk}T_{(iaa)} + \delta_{ki}T_{(jaa)}] + \frac{1}{3}[(T_{ijk} - T_{jik}) + (T_{ijk} - T_{kij})] \tag{C.8}$$

The first term is an irreducible representation of order 3, the second group of terms 3 irreducible representations of order 1, and the last two terms two irreducible representations of order 2. Upon noting that

$$T_{(i\ell\ell)} = \frac{1}{3}(T_i + 2U_i), \tag{C.9}$$

with the definitions (38), (C.8) reduces to (37).

For the fourth-order tensor we have

$$T_{ijk\ell} = \widehat{T}_{ijk\ell} - \sum_{k=1}^{2} a_k^4 \delta_{(i_1 i_2 \ldots} \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \ldots i_4) a_1 a_1 \ldots a_k a_k} + \frac{1}{4}[(T_{ijk\ell} - T_{jik\ell}) + (T_{ijk\ell} - T_{kji\ell}) + (T_{ijk\ell} - T_{\ell kji})]. \tag{C.10}$$

with

$$\sum_{k=1}^{2} a_k^4 \delta_{(i_1 i_2 \ldots} \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \ldots i_4) a_1 a_1 \ldots a_k a_k} = -\frac{1}{7}\left[\delta_{ij} T_{(k\ell aa)} + \delta_{ik} T_{(j\ell aa)} + \delta_{i\ell} T_{(jkaa)} + \delta_{jk} T_{(i\ell aa)} + \delta_{j\ell} T_{(ikaa)} + \delta_{k\ell} T_{(ijaa)}\right] + \frac{1}{35}(\delta_{ij}\delta_{k\ell} + \delta_{jk}\delta_{i\ell} + \delta_{i\ell}\delta_{jk}) T_{(aabb)}. \tag{C.11}$$

It is necessary to separate out the trace of the tensors appearing here, which we do by noting that

$$T_{(ijk\ell)} = \frac{1}{4}[T_{ijk\ell} + T_{jk\ell i} + T_{k\ell ij} + T_{\ell ijk}] \tag{C.12}$$

so that

$$T_{(ijaa)} = \frac{1}{4}[T_{ijaa} + T_{jaai} + T_{aaij} + T_{aija}] = \frac{1}{4}[T_{ij} + T_{ji} + 2U_{ij}] \tag{C.13}$$

and $T_{(aabb)} = U_0$. With the definition

$$\widehat{T}_{(ijaa)} = T_{(ijaa)} - \frac{1}{3}\delta_{ij} U_0 \tag{C.14}$$

(C.11) becomes

$$\sum_{k=1}^{2} a_k^4 \delta_{(i_1 i_2 \ldots} \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \ldots i_4) a_1 a_1 \ldots a_k a_k} = -\frac{1}{7}[\delta_{ij} \widehat{T}_{(k\ell aa)} + \delta_{ik} \widehat{T}_{(j\ell aa)} + \delta_{i\ell} \widehat{T}_{(jkaa)} + \delta_{jk} \widehat{T}_{(i\ell aa)} + \delta_{j\ell} \widehat{T}_{(ikaa)} + \delta_{k\ell} \widehat{T}_{(ijaa)}] - \frac{1}{15}\left(\delta_{ij}\delta_{k\ell} + \delta_{jk}\delta_{i\ell} + \delta_{i\ell}\delta_{jk}\right) U_0 \tag{C.15}$$

which, when inserted into (C.10), gives the earlier result (41).

Appendix D. Viscous contribution to the tensors U

In the case of a Newtonian fluid, the contribution of the viscous part of the stress to the tensors U defined in (31) is the integral of

$$\mu_C \mathbf{n} \cdot (\nabla \mathbf{u}_C + \nabla \mathbf{u}_C^T) \cdot \mathbf{n} = 2\mu_C \mathbf{n} \cdot (\nabla \mathbf{u}_C) \cdot \mathbf{n} \tag{D.1}$$

over the particle surface. Let us adopt a (generally non-inertial) reference frame in which the particle is at rest by writing

$$\mathbf{u}_C = \mathbf{w} + \mathbf{\Omega} \times \mathbf{x} + \mathbf{u}' \tag{D.2}$$

in which $\mathbf{u}'$ is the velocity in the new frame and $\mathbf{x}$ is measured from the particle center. Then the Navier–Stokes equation may be written as

$$-\nabla\tilde{p} + \mu_C \nabla^2 \tilde{\mathbf{u}} = \mathbf{f} \tag{D.3}$$

where we have introduced the definitions

$$\tilde{p} = p - \frac{1}{2}\rho(\mathbf{\Omega} \times \mathbf{x})^2 + \rho(\mathbf{g} - \dot{\mathbf{w}}) \cdot \mathbf{x}, \tag{D.4}$$

$$\tilde{\mathbf{u}} = \mathbf{u}' - \rho_C \frac{r^5 - a^5}{10\mu_C r^3} \dot{\mathbf{\Omega}} \times \mathbf{x}, \tag{D.5}$$

$$\mathbf{f} = \rho\left(\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla\mathbf{u}' + 2\mathbf{\Omega} \times \mathbf{u}'\right). \tag{D.6}$$

It will be noted that, with these definitions, both $\mathbf{u}'$ and $\tilde{\mathbf{u}}$ vanish at the particle surface due to the no-slip condition, from which it follows in particular that $\mathbf{f} = 0$ on the particle surface as well.

It is easy to show that, at the particle surface,

$$\mathbf{n} \cdot (\nabla\mathbf{u}_C) \cdot \mathbf{n}|_{r=a} = \mathbf{n} \cdot (\nabla\tilde{\mathbf{u}}) \cdot \mathbf{n}|_{r=a}. \tag{D.7}$$

Eq. (D.3) shows that $\tilde{\mathbf{u}}$ formally satisfies the non-homogeneous Stokes equation, the solution of which may be broken up into the sum of the general solution of the homogeneous equation, $(p^h, \mathbf{u}^h)$, and of a particular solution of the non-homogeneous equation, $(p^p, \mathbf{u}^p)$, both of which may be taken to vanish at the particle surface. By using Lamb's general solution of the Stokes equation, it can be shown by a direct calculation that the homogeneous part gives a zero contribution to (D.7). That the same must be true for the particular solution follows from the fact that, since $\mathbf{f}$ vanishes as one approaches the particle surface, $\nabla\mathbf{u}^p$ must vanish even faster since $\mathbf{u}^p$ solves (D.3).

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